Talk at NEU

"Quantum toroidal and affine Yangian of $\mathfrak{gl}_1$"

04/07/2014

Plan:

- Motivation (repr. theory)
- Key definitions and geom. actions
- Degeneration
- Shuffle algebras
- Commutative subalgebras
- Some representations.
- Analogue of Sachin-Valerio homom. of completions

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① Recall the classical action of the Heisenberg algebra on the sum of cohomologies of the $(\mathbb{A}^2)^{[n]}$ via Nakajima-Grojnowski operator.

Q1: Do we have an action of "something bigger" on that space, which might be of interest?

Q2: What about the generalization of the above construction to the case of K-theory?

Goal: Our goal for today is to introduce appropriate algebras acting both on the cohom./K-theory of $(\mathbb{A}^2)^{[n]}$ (sum over n). They will contain a Heisenberg subalgebra that is of interest But they are also of interest purely for algebraists

History: In the case of K-theory, those algebras appeared independently in [Schiffmann-Vasserot '09] and [Feigin-Tsymbaliuk '09]

In the case of cohomology there is a big theory elaborated by Maulik-Okounkov. However, they never provide an algebraic description of the algebra itself.

Quantum toroidal of gl_1

Let q_1, q_2, q_3 be complex parameters satisfying $q_1 \cdot q_2 \cdot q_3 = 1$, $q_i \neq 1$.
Algebra $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ is generated by $\{e_i, f_i, \psi_j^{\pm}, (\psi_0^{\pm})^{-1} \mid i \in \mathbb{Z}, j \geq 0\}$ with the following defining relations:

(T0) $\psi_0^{\pm} \cdot (\psi_0^{\pm})^{-1} = (\psi_0^{\pm})^{-1} \cdot \psi_0^{\pm} = 1$, $\psi_i^{\varepsilon} \cdot \psi_j^{\varepsilon'} = \psi_j^{\varepsilon'} \cdot \psi_i^{\varepsilon}$ $\quad \varepsilon, \varepsilon' \in \{\pm\}$.

(T1) $e(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) = -e(w)e(z)(w-q_1z)(w-q_2z)(w-q_3z)$

(T2) $f(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) = -f(w)f(z)(z-q_1w)(z-q_2w)(z-q_3w)$

(T3) $[e(z), f(w)] = \frac{1}{(1-q_1)(1-q_2)(1-q_3)} \delta\left(\frac{z}{w}\right)(\psi^+(w) - \psi^-(z))$

(T4) $\psi^{\pm}(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) = -e(w)\psi^{\pm}(z)(w-q_1z)(w-q_2z)(w-q_3z)$

(T5) $\psi^{\pm}(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) = -f(w)\psi^{\pm}(z)(z-q_1w)(z-q_2w)(z-q_3w)$

(T6) $\mathrm{Sym}_{S_3}[e_{i_1}, [e_{i_2+1}, e_{i_3-1}]] = 0$, $\mathrm{Sym}_{S_3}[f_{i_1}, [f_{i_2+1}, f_{i_3-1}]] = 0$

where $e(z) := \sum_{i \in \mathbb{Z}} e_i z^{-i}$, $f(z) = \sum_{i \in \mathbb{Z}} f_i z^{-i}$, $\psi^{\pm}(z) = \sum_{j \geq 0} \psi_j^{\pm} z^{\mp j}$, $\delta(z) = \sum_{i \in \mathbb{Z}} z^i$.

Affine Yangian of gl_1

The affine Yangian of gl_1 depends on 3 parameters h_1, h_2, h_3 s.t. $h_1 + h_2 + h_3 = 0$
It is generated by $\{e_j, f_j, \psi_j \mid j \geq 0\}$ with the following defin. rel-s:

(Y0) $[\psi_i, \psi_j] = 0$

(Y1) $[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]) - \sigma_3\{e_i, e_j\} = 0$

(Y2) $[f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] + \sigma_2([f_{i+1}, f_j] - [f_i, f_{j+1}]) + \sigma_3\{f_i, f_j\} = 0$

(Y3) $[e_i, f_j] = \psi_{i+j}$ $\quad$ (Y4', 5') $[\psi_2, e_j] = 2e_j$, $[\psi_2, f_j] = -2f_j$, $[\psi_1, e_j] = [\psi_0, e_j] = [\psi_1, f_j] = [\psi_0, f_j] = 0$

(Y4) $[\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] + \sigma_2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) - \sigma_3\{\psi_i, e_j\} = 0$

(Y5) $[\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] + \sigma_2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) + \sigma_3\{\psi_i, f_j\} = 0$

(Y6) $\mathrm{Sym}_{S_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0$, $\mathrm{Sym}_{S_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0$

Rmks: (1) The above presentations are similar to the classical quantum loop algebra and the Yangian of $\mathfrak{g}$. Moreover, their def. rel-s are related to each other in exactly the same fashion.
(2) What is quite nontrivial to see is that we need the following boundary rel-s: $[\psi_2, e_i] = 2e_i$, $[\psi_2, f_i] = -2f_i$, ψ_0, ψ_1 - central.
(3) Using (T4, T5) and (Y4, Y5) cubic rels (T6), (Y6) is sufficient to require for $i_1 = i_2 = i_3 = 0$

③ Geometric action I

$X=\mathbb{A}^2 \rightsquigarrow X^{[n]}$ - Hilbert scheme of n points $\rightsquigarrow P[i] \subset \coprod_n X^{[n]}\times X^{[n+1]}$

Nakajima - Grojnowski correspondence

Unlike general $P[i]$, the correspondences $P[\pm 1]$-smooth.

$$\begin{tikzcd} & P^n_{[1]}\subset X^{[n]}\times X^{[n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X^{[n]} & & X^{[n+1]} \end{tikzcd}$$

Notation:
- p,q - natural projections from $P[1]$
- L - natural line bundle on $P[1]$
- $\mathbb{T}:=\mathbb{C}^*\times\mathbb{C}^*$
- $\mathcal{F}$ - tautological v. bundle on $X^{[n]}$.

$\mathbb{T}\curvearrowright X \rightsquigarrow \mathbb{T}\curvearrowright X^{[n]} \rightsquigarrow (X^{[n]})^{\mathbb{T}} = \{J_\lambda\}_{\lambda\text{-Young diagram of size } n}$.

$(t_1,t_2)\cdot(x,y)=(t_1x,t_2y)$

$J_\lambda = \mathbb{C}[x,y]\cdot(\mathbb{C}x^{\lambda_1}y^0\oplus\mathbb{C}x^{\lambda_2}y^1\oplus\cdots)$

(a) Define $M:=\bigoplus_n \underbrace{K^{\mathbb{T}}(X^{[n]})_{loc}}_{M_n}$ - the sum of localized equivariant K-gps.

Finally consider $a(z):=\Lambda^\bullet_{-1/z}(\mathcal{F})=\sum_{i\ge 0}[\Lambda^i\mathcal{F}]\cdot(-1/z)^i$.

$$c(z):=\frac{a(zt_1)a(zt_2)a(zt_3)}{a(zt_1^{-1})a(zt_2^{-1})a(zt_3^{-1})},$$ where t_1,t_2 - natural coord. on $\mathbb{T}$, $t_3:=1/t_1t_2$.

Consider the following operators on M:

$$\boxed{\begin{aligned} e_i &:= q_*(L^{\otimes i}\otimes p^*): M_n\longrightarrow M_{n+1}\\ f_i &:= p_*(L^{\otimes(i-1)}\otimes q^*): M_{n+1}\longrightarrow M_n\\ \psi^\pm(z)|_{M_n} &:= \left(-\frac{1-t_3z^{-1}}{1-z^{-1}}c(z)\right)^\pm \in M_n[z^{\mp 1}]\end{aligned}} \quad (1)$$

$\pm$ denotes expansion in $z^{\mp 1}$,

Theorem 1: Formulas (1) define an action of $\ddot{U}_{t_1,t_2,t_3}(\mathfrak{gl}_1)$ on M.

b) Define $V:=\bigoplus_n H^*_{\mathbb{T}}(X^{[n]})_{loc}$ - the sum of localized equiv. cohom. gps.

Consider $$C(z):=\left(\frac{ch(\mathcal{F}t_1^{-1},-1/z)\,ch(\mathcal{F}t_2^{-1},-1/z)\,ch(\mathcal{F}t_3^{-1},-1/z)}{ch(\mathcal{F}t_1,-1/z)\,ch(\mathcal{F}t_2,-1/z)\,ch(\mathcal{F}t_3,-1/z)}\right)^+$$

Note that V is a module over $\mathbb{C}(s_1,s_2)$, where s_1,s_2 - natural basis of $\mathfrak{t}=\mathrm{Lie}\,\mathbb{T}$ and $s_3:=-s_1-s_2$.

Consider the following operators on V:

$$\boxed{\begin{aligned} e_j &:= q_*(c_1(L)^j\cdot p^*): V_n\longrightarrow V_{n+1}\\ f_j &:= p_*(c_1(L)^j\cdot q^*): V_{n+1}\longrightarrow V_n\\ \psi(z)|_{V_n} &:= ((1-s_3/z)\,C(z))^+\end{aligned}} \quad (2)$$

Theorem 2: Formulas (2) define an action of $\ddot{Y}_{s_1,s_2,s_3}(\mathfrak{gl}_1)$ on V.

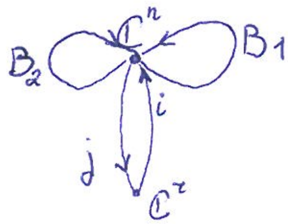
3' Geometric action II

There is a natural generalization of previous 2 results to the higher rank. Recall that the Hilbert scheme of points $(\mathbb{A}^2)^{[n]}$ is actually the first member of the series of moduli spaces $M(r,n)$, called Gieseker moduli space.

$\mathcal{M}(r,n) = \{(E,\Phi)\}/_{\underset{isom}{\sim}}$, where
- E: torsion free sheaf on $\mathbb{P}^2$, $rk(E)=r$, $c_2(E)=n$
- E - loc. free in the nbhd of $l_\infty = \{(0:*:*)\} \subset \mathbb{P}^2$
- $\Phi: E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{|l_\infty}^{\oplus r}$ - "framing at ∞".

This space has an alternative quiver description:

$$M(r,n) = \mathcal{M}(r,n)/GL_n(\mathbb{C}), \quad \mathcal{M}(r,n) := \{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\}^{s}$$



stability condition means there is no proper subspace $S \subsetneq \mathbb{C}^n$ containing $Im(i)$ and B_1, B_2-invariant.

Let $\mathbb{T}_r := (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r$. This torus acts on $M(r,n)$, where $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$, while $(\mathbb{C}^*)^r$ acts by changing the framing at infinity.

The locus of $\mathbb{T}_r$-fixed points in $M(r,n)$ is finite and is parametrized by the r-partitions $\bar{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of n; we denote the fixed pt by $\xi_{\bar{\lambda}}$.

It is possible to define an analogue of $P[1]$, which is called the <u>Hecke correspondence</u> $M(r; n, n+1) \subset M(r,n) \times M(r,n+1)$. It is a smooth variety of dimension $2rn + r + 1$. Call projections to $M(r,n)$, $M(r,n+1)$ by p_r, q_r.

Let L_r be the tautological line bundle on $M(r; n, n+1)$

$\mathcal{F}_r$ — " — vector bundle on $M(r,n)$.

Consider $M^r := \bigoplus_n K^{\mathbb{T}_r}(M(r,n))_{loc}$, $V^r := \bigoplus_n H^*_{\mathbb{T}_r}(M(r,n)_{loc})$.

Theorem 3: The following formulas define an action of $\ddot{U}_{t_1,t_2,t_3}(\mathfrak{gl}_1) \curvearrowright M^r$

$$e_i := q_{r*}(p_r^* \otimes L_r^{\otimes i}), \quad f_i := p_{r*}(L_r^{\otimes(i-r)} \otimes q_r^*), \quad \psi^{\pm}(z)|_{M_n^r} := (? \cdot C_r(z))^{\pm},$$

where $C_r(z)$ is defined completely analogously to $C(z)$, while "?" is the coefficient equal to $(-1)^r t_1 t_2 x_1 \ldots x_r \cdot \prod_{a=1}^{r} \frac{1 - t_1 t_2 x_a z}{1 - x_a z}$.

Theorem 4: The following f-las define an action of $\ddot{Y}_{s_1,s_2,s_3}(\mathfrak{gl}_1) \curvearrowright V^r$

$$e_j := q_{r*}(c_1(L_r)^j \cdot p_r^*), \quad f_j := p_{r*}(c_1(L_r)^j \cdot q_r^*), \quad \psi(z)|_{V_n^r} := (? \cdot C_r(z))^+,$$

where $C_r(z)$ is defined as $C(z)$, while $? = \prod_{a=1}^{r} \frac{z + x_a - s_3}{z + x_a}$

④ Degeneration

Let us now describe algebras $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ in their limit as $q_3 \to 1$ or $h_3 \to 0$.

(a) Algebras of difference operators on $\mathbb{C}$, $\mathbb{C}^*$.

Let $\mathcal{D}_q$ be the associative algebra generated by $Z^{\pm 1}$, $D^{\pm 1}$ with the defining rel-n $DZ = qZD$ (there are obvious rel-s like $Z \cdot Z^{-1} = 1$ etc).

We will view $\mathcal{D}_q$ as a Lie algebra w.r.t. commutator bracket.

It is known that the second cohomology group of this Lie alg. is 2-dim.

In particular, the following 2-cocycle defines a nontrivial central extension $\bar{\mathcal{D}}_q := \mathcal{D}_q \oplus \mathbb{C} \cdot c_{\mathcal{D}}$. We also use $\bar{\mathcal{D}}_q$ to denote a Lie subalg. spanned by $c_{\mathcal{D}}$, $Z^i D^j$ $_{(i,j)\neq(0,0)}$

2-cocycle: $\phi_{\mathcal{D}}(Z^a D^j, Z^b D^{-j'}) = \begin{cases} 0, & j \neq j' \text{ or } j = j' = 0 \\ \sum_{i=-j}^{-1} q^{ai + b(j+i)}, & j = j' > 0 \\ \text{—''—} & , j = j' < 0 \end{cases}$

Let $\mathcal{D}_h$ be the assoc. alg., generated by $x, \partial^{\pm 1}$ with the def. rel-n
$\partial \cdot x = (x+h)\partial$.

We will view $\mathcal{D}_h$ as a Lie algebra. It has a 1-dim 2^{nd} cohomology.

In particular, the 2-cocycle

$$\phi_{\mathcal{D}}(f(x)\partial^r, g(x)\partial^{-s}) = \begin{cases} 0, & r \neq s \text{ or } r = s = 0 \\ \sum_{\ell=-r}^{-1} f(\ell h) g((\ell+r)h), & r = s > 0 \\ \text{—''—} & , r = s < 0 \end{cases}$$

defines a nontrivial central extension $\bar{\mathcal{D}}_h = \mathcal{D}_h \oplus \mathbb{C} \cdot c_{\mathcal{D}}$.

(b) Algebra $\ddot{Y}'_{h_1,-h_1,0}(\mathfrak{gl}_1)$.

First we rescale el-s ψ_i by $-h_1 h_2$: this changes $[\psi_2, e_j] = 2e_j$ to $[\psi_2, e_j] = -2h_1 h_2 e_j$.

Benefit: When h_1, h_2, h_3 are viewed formal, then $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is $\mathbb{Z}_+$-graded (unlike $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$) with $\deg(e_j) = \deg(f_j) = \deg(\psi_j) = j$, $\deg(h_i) = 1$.

Theorem 5: The following f-las extend to an isomorphism $\ddot{Y}'_{h_1,-h_1,0}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathcal{D}}_{h_1})$

$$\boxed{e_j \mapsto x^j \partial, \quad f_j \mapsto -\partial^{-1} x^j, \quad \psi_j \mapsto (x-h)^j - x^j - (-h)^j c_{\mathcal{D}}}$$

(c) Algebra $\ddot{U}'_{q,q^{-1},1}(\mathfrak{gl}_1)$.

You can't naively take $q_3 = 1$ as this will spoil (T3). Let $h_3 = \ln q_3$.

Instead we write $\psi^{\pm}(z) = \exp(\pm \frac{h_3}{2} \varkappa) \cdot \exp(\pm(1-q_3) \sum_{\pm m > 0} H_m z^{-m})$, $e(z) \rightsquigarrow (1-q_1)e(z)$, $f(z) \rightsquigarrow (1-q_2)f(z)$

Then rel-n (T3) makes sense in the limit $h_3 \to 0$.

Theorem 6: For $q \neq \sqrt{1}$, the following f-las extend to an isom. $\ddot{U}'_{q,q^{-1},1}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\bar{\mathcal{D}}'_q)$

$$\boxed{e_i \mapsto Z^i D, \quad f_i \mapsto -D^{-1} Z^i, \quad H_k \mapsto (q^{-k}-1) Z^k - q^{-k} c_{\mathcal{D}}, \quad \varkappa \mapsto c_{\mathcal{D}}}$$

⑤ Shuffle algebras

The notion of shuffle algebras was introduced by Feigin–Odesskii in late 90's. They considered elliptic case, while we need the trig/rational one.

$$\mathbb{S} := \bigoplus_{n\geq 0} \mathbb{S}_n \,, \quad \text{where} \quad \mathbb{S}_n = \left\{ \frac{f(x_1, \ldots x_n)}{\underbrace{\Delta(x_1, \ldots x_n)^2}_{\text{Vandermonde determinant}}} \,\middle|\, f\in \mathbb{C}[x_1^{\pm 1}, \ldots x_n^{\pm 1}]^{S_n} \right\}$$

Define the star-product on $\mathbb{S}$ by

$$(F*G)(x_1, \ldots x_{n+m}) = \mathrm{Sym}\Big(F(x_1, \ldots x_n)\, G(x_{n+1}, \ldots x_{n+m}) \prod_{\substack{k\leq n\\ \ell>n}} \lambda(x_k, x_\ell)\Big),$$

$$\lambda(x,y) = \frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}$$

This makes $\mathbb{S}$ into a unital algebra. But it is <u>HUGE</u>!

We say that $\frac{f(x_1, \ldots x_n)}{\Delta(x_1, \ldots x_n)^2}$ satisfies the wheel conditions if for any $\{x_1, \ldots x_n\}$ s.t. $x_1/x_2 = q_i$, $x_2/x_3 = q_j$ $(i\neq j)$, any $x_4, \ldots x_n$ $f(-)=0$.

It is easy to see that the space $S = \oplus S_n$, $S_n \subset \mathbb{S}_n$ of such f-s is closed w.r.t. $*$.

The following fact was conjectured in [FT] and proved by Negut

Theorem 7 (Negut '12): Algebra S is generated by S_1.

The connection b/w S and $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ is as follows.

Let $\ddot{U}^+$ be the subalgebra of $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ generated by e_i. As an abstract algebra, it is gen. by (e_i) with def. rel-s (T1, T6)

We have a natural homom. $\ddot{U}^+ \longrightarrow S$ $\quad e_i \mapsto x_1^i \in S_1$.

As a consequence of the above thm and some results of Schiffmann:

Corollary: $\ddot{U}^+ \xrightarrow{\sim} S$ is an isom.

Moreover $\ddot{U}_{q_1,q_2,q_3}(gl_1) \simeq \mathcal{D}(S)$ – Drinfeld double of S.

Similar construction works also for the case

$$\lambda(x,y) = \frac{(x-y-h_1)(x-y-h_2)(x-y-h_3)}{(x-y)^3}$$

Need to consider $f\in \mathbb{C}[x_1, \ldots x_n]^{S_n}$ (not $\mathbb{C}[x_1^{\pm1}, \ldots x_n^{\pm1}]^{S_n}$).

Then all results from above generalize to yield $\ddot{Y}^+ \simeq S$.

⑥ Commutative subalgebras and Heisenberg alg. action

Recall that our initial goal was to obtain a natural Heisenberg action on M (generalizing the Nakajima - Grojnowski for V).

So far we have constructed $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \curvearrowright M$, but all the operators e_i shift degree only by 1, i.e. $e_i: M_n \to M_{n+1}$. While our Heisenberg operator a_k should shift by k.

It turns out that while $\{a_k\}$ are quite complicated el-s of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ they have a nice explicit description in shuffle terms.

Let us consider a $\mathbb{Z}_+$-graded subspace $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ of S, defined by

$$\mathcal{A}_n = \{F \in S_n \mid \partial^{(\infty,k)} F = \partial^{(0,k)} F \ \forall\, 0 \leq k \leq n\}, \text{ where}$$

$$\partial^{(0,k)} F = \lim_{\xi \to 0} F(x_1, \ldots x_{n-k}, \xi \cdot x_{n-k+1}, \ldots \xi \cdot x_n)$$

$$\partial^{(\infty,k)} F = \lim_{\xi \to \infty} F(x_1, \ldots x_{n-k}, \xi \cdot x_{n-k+1}, \ldots \xi \cdot x_n).$$

The following result is due to Feigin - Hashizume - Hoshino - Shiraishi - Yanagida

<u>Theorem 8 ([FHHSY '09])</u>: The subspace $\mathcal{A}^m$ is $*$-commutative. Moreover, it is isomorphic to $\mathbb{C}[K_1, K_2, K_3, \ldots]$,

$$K_1(x) := x_1^0, \quad K_n(x_1; x_2; \ldots; x_n) := \prod_{1\leq i<j\leq n} \frac{(x_i - q_1 x_j)(x_j - q_1 x_i)}{(x_i - x_j)^2}.$$

Note that el-s K_i are explicit, but their expression via $(S_1, *)$ is completely nontrivial. However, we would like to have an alternative set of generators expressed via S_1.

<u>Lemma</u>: $\mathcal{A}^m$ is a free comm. alg. in $\{L_j\}_{j\geq 1}$, where

$$L_1(x_1) = x_1^0, \quad L_n(x_1, \ldots x_n) = \underbrace{[x^1, [x^0, [\ldots, [x^0, x^{-1}]\ldots]}_{n\text{-commutator}}.$$

Similar results apply to the "additive case" as well:

<u>Theorem 9</u>: (a) The following el-s are $*$-commutative and alg. indep.

$$K_1^a(x) = x^0, \quad K_n^a(x_1, \ldots x_n) := \prod_{i<j} \frac{(x_i - x_j - h_1)(x_j - x_i - h_1)}{(x_i - x_j)^2}$$

(b) An alternative set of generators for the corresp alg. is

$$L_1^a(x_1) = x_1^0, \quad L_n^a(x_1, \ldots x_n) = \underbrace{[x^0, [x^0, \ldots [x^0, x^{n-1}]\ldots]}_{n\text{-commutator}}.$$

6' Recovering the Heisenberg action on M

- $\bigoplus_n K^T(X^{[n]})_{loc} =: M = \bigoplus_\lambda \mathbb{C}(t_1, t_2) \cdot [\lambda]$ ← Fixed point decomposition.
- $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) = \mathcal{D}(S) \curvearrowright M \rightsquigarrow K_n \curvearrowright M.$

Key Observation ([FT]): There are constant c_λ, s.t. the isomorphism of vector spaces $M \xrightarrow{\sim} \Lambda$ - the ring of symm. polyn. in $x_1, x_2, \ldots$

$[\lambda] \longmapsto c_\lambda \cdot P_\lambda$ - Macdonald polynomial

intertwines K_n with operators $\cdot e_n : \Lambda \longrightarrow \Lambda$

Upshot: Therefore taking K_n and the opposite $K_{-n} \in S^{op}$ we get that they act as "vertex-type" operators of the Heisenberg on M.

Here we use f-la $1 + \sum_{i>0} e_i z^i = \exp\left(\sum_{j>0} \frac{(-1)^{j-1}}{j} p_j z^j\right)$

and $\{p_j\}$ are exactly the Heisenberg operators $\curvearrowright \Lambda$.

Additive case: In the "cohomological case" operators K^a also provide the "vertex-type" operators of Heisenberg $\curvearrowright V$. Moreover, this Heisenberg is the same as the classical action of Nakajima - Grojnowski.

(in the identification $V \xrightarrow{\sim} \Lambda$ $\quad [\lambda] \mapsto c^a_\lambda \cdot J_\lambda$ - Jack pol.)

This has been observed by Li-Qin-Wang '04.

7 Representations of $\ddot{U}(\mathfrak{gl}_1)$ & $\ddot{Y}(\mathfrak{gl}_1)$

The rep. theory of those algebras has not been classified yet. However, there are interesting series for $\ddot{U}(\mathfrak{gl}_1)$ studied by Feigin - Feigin - Jimbo - Miwa - Mukhin in multiple papers.

1 Vector representations $V(u)$

There is a 1-parameter family of $\ddot{U}(\mathfrak{gl}_1)$-rep-s on the space w/ basis $\{[u]_j\}_{j\in\mathbb{Z}}$ given by

$$\boxed{e(z)[u]_i = \frac{1}{1-q_1}\delta\left(\frac{q_1^i u}{z}\right)[u]_{i+1},\quad f(z)[u]_i = \frac{-1}{1-q_1^{-1}}\delta\left(\frac{q_1^{i-1}u}{z}\right)[u]_{i-1},\quad \psi^\pm(z)[u]_i = \left(\frac{(z-q_1^i q_2 u)(z-q_1^i q_3 u)}{(z-q_1^i u)(z-q_1^{i-1}u)}\right)^\pm [u]_i}$$

2 Fock modules $F(u)$

There is also a 1-parameter family of modules $F(u)$. They are obtained from $V(u)$ by taking the $\frac{\infty}{2}$-wedge construction of

$$V(u)\otimes V(uq_3^{-1})\otimes V(uq_3^{-2})\otimes\ldots$$

To define $\otimes$ we use the following "formal coproduct"

$$\boxed{\Delta(e(z)) = e(z)\otimes 1 + \psi^-(z)\otimes e(z),\quad \Delta(f(z)) = f(z)\otimes\psi^+(z) + 1\otimes f(z),\quad \Delta(\psi^\pm(z)) = \psi^\pm(z)\otimes\psi^\pm(z)}$$

Rmk: This is not a "decent" coproduct since it has ∞ many summands. However, in all cases of interest it works since the matrix coeff. for $e(z)$ are usually $?\cdot\delta(\mu/z)$ and we can use the f-la

$$(*)\qquad \gamma(z)\cdot\delta(\mu/z) = \gamma(\mu)\cdot\delta(\mu/z)\qquad\text{for any rational f-n } \gamma.$$

3 Resonance conditions

Under some resonance conditions on $u_1,\ldots,u_n$ the above f-la $(*)$ doesn't provide an action of $\ddot{U}(\mathfrak{gl}_1)$ say on $V(u_1)\otimes\ldots\otimes V(u_n)$ or $F(u_1)\otimes\ldots\otimes F(u_n)$.

In particular, in [FFJMM] the authors introduced the natural action of $\ddot{U}(\mathfrak{gl}_1)$ on the space w/ basis consisting of (k,r)-admissible partit.

$$\lambda = (\lambda_1,\ldots,\lambda_N) \text{ is } (k,r)\text{-admissible} \iff \lambda_i - \lambda_{i+k} \geq r \quad \forall\, i\leq N-k.$$

Moreover, one can also consider their limit as $N\to\infty$.

⑦ Representations of $\ddot{U}(\mathfrak{gl}_1)$, $\ddot{Y}(\mathfrak{gl}_1)$

It was observed in [FFJMM1] that $F(1) \xrightarrow{\sim} M$

$$|\lambda\rangle \mapsto c_\lambda \cdot [\lambda] \qquad c_\lambda \in \mathbb{C}(q_1, q_2).$$

Moreover, we have the following result:

Theorem 10 : There exist a unique collection of constants $c_{\bar{\lambda}} \in \mathbb{C}(q_1, q_2, x_1, \dots, x_2)$, $c_{\bar{\emptyset}} = 1$ s.t. the map

$$M^2 \xrightarrow{\sim} F(x_1) \otimes \dots \otimes F(x_2) \quad \text{is an isom. of}$$
$$[\bar{\lambda}] \longmapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \dots \otimes |\lambda^2\rangle \quad \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)\text{-modules}.$$

It turns out that there is exactly the same theory for $\ddot{Y}(\mathfrak{gl}_1)$. The Δ on $\ddot{Y}(\mathfrak{gl}_1)$ is analogous (it makes sense only on the subcategory of admissible representations).

In particular $\ddot{Y}(\mathfrak{gl}_1)$ acts on the same spaces as above.

The only changes required are:

$$\delta\left(\frac{q_1^i q_2^j q_3^k u}{z}\right) \rightsquigarrow \frac{1}{z}\delta^+((ih_1 + jh_2 + kh_3 + u)/z), \quad 1 - \frac{q_1^i q_2^j q_3^k u}{z} \rightsquigarrow ih_1 + jh_2 + kh_3 + u - z.$$

where $\delta^+(z) = 1 + z + z^2 + \dots$

The analogue of Thm 10 also holds and the corresponding statement played a crucial role in Maulik–Okounkov' study of quantum cohomology of Hilbert schemes $(\mathbb{A}^2)^{[n]}$.

⑧ Relation b/w $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \leftrightarrow \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$

In the recent papers Gautam - Toledano Laredo explained a similar relation b/w $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$.

They constructed a homom.

$$\Phi : U_q(L\mathfrak{g}) \longrightarrow \widehat{Y_\hbar(\mathfrak{g})} \qquad q = e^\hbar.$$

which has the form

$$U^0(L\mathfrak{g}) \longrightarrow \widehat{Y^0_\hbar}(\mathfrak{g})$$

$$U^+(L\mathfrak{g}) \longrightarrow \widehat{Y^{\geq 0}_\hbar}(\mathfrak{g})$$

$$U^-(L\mathfrak{g}) \longrightarrow \widehat{Y^{\leq 0}_\hbar}(\mathfrak{g})$$

They also proved that it induces an isom. of completions

$$\boxed{\widehat{\Phi} : \widehat{U_q(L\mathfrak{g})} \xrightarrow{\sim} \widehat{Y_\hbar(\mathfrak{g})}},$$

where the target is completed w.r.t. a $\mathbb{Z}_+$-grading, while the source is completed w.r.t. powers of an ideal J, which is define as the kernel

$$U_q(L\mathfrak{g}) \xrightarrow{q \to 1} U(L\mathfrak{g}) \xrightarrow{z \to 0} U(\mathfrak{g}).$$

In particular, $\mathrm{gr}\,\Phi$ establishes $Y_\hbar(\mathfrak{g})$ as the associated graded of $U_q(L\mathfrak{g})$.

This also explains the similarity of the theories of repr-s for algebras $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$

__Rmk__: On the classical level (i.e. as $\hbar \to 0$) $\widehat{\Phi}$ is given by

$$\widehat{U(\mathfrak{g}[z,z^{-1}])} \longrightarrow \widehat{U}(\mathfrak{g}[w])$$

$$A \cdot z^n \longmapsto A \cdot \exp(n \cdot w).$$

⑧ Relation b/w $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \leftrightarrow \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$

It was already mentioned that we better consider the algebras $\ddot{U}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ and $\ddot{Y}'_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ (this makes the latter alg. graded, while the former has a limit as $h_3 \to 0$)

However, they are no longer S_3-invariant as the original alg-s.

Then analogous arguments as those in [GTL] provide a homom.

$$\Upsilon : \ddot{U}'_{h_1,h_2,h_3}(\mathfrak{gl}_1) \longrightarrow \widehat{\ddot{Y}'}_{h_1,h_2,h_3}(\mathfrak{gl}_1).$$

However it is not an isom. on the level of completions, since already in the classical case ($h_3 \to 0$) it is given by

$$\bar{\partial}'_h \longrightarrow \widehat{\bar{\mathcal{D}}_h}$$

$$Z^i D^j \longmapsto \exp(i \cdot x) \cdot \partial^j \quad , \quad c_\partial \longmapsto c_{\mathcal{D}}$$

But we only have the isomorphism

$\widehat{\bar{\partial}_h} \xrightarrow{\sim} \widehat{\bar{\mathcal{D}}_h}$, where the completions are taken w.r.t. the powers of ideals (Z, h), (x, h).

However, $\bar{\partial}'_h \subset \bar{\partial}_h$ doesn't include the el-t $Z^0 D^0$.

So we get that $\hat{\Upsilon}$ is "almost an isomorphism" up to central extensions.

<u>Remark</u>: (1) One actually show that both algebras are flat h_3-deformations of the respective quotients $U(\bar{\partial}'_h)$ and $U(\bar{\mathcal{D}}_h)$.

(2) Finally Υ is compatible with M^r, V^r.
In other words the map

$$ch_r : M^r \longrightarrow \widehat{V^r}$$
$$[\lambda] \longmapsto [\lambda]$$
$$t_i \longmapsto \exp(s_i)$$
$$X_j \longmapsto \exp(x_j)$$

satisfies the property

$$ch_r(X \cdot v) = \Upsilon(X) \cdot ch_r(v).$$

(one needs to be careful when defining M^r, V^r for formal parameters q_i, h_i)